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it is necessary that the line ac have a positive slope greater than unity but not infinite.

Suppose the slopes of ab and ac both positive and greater than unity. If the slope of ab [or ac] is less than that of ac [or ab] then B [or C] may win over both A and C [or B] by choosing as weights the coördinates of any point within the region PQR_1 [or PQR_2]. From Fig. 2 (or Fig. 3) in order that C [or B] may win over B [or C] and A it is necessary that the line bc have a positive slope greater than unity but not infinite and also greater than the slope of ac [or ab].

Thus, in order that either one of B or C at will may win over the other two candidates, A having the greatest number of votes for first choice, it is necessary that each of the three lines ab , ac , bc have a positive slope greater than unity but not infinite and that of these three lines the slope of bc be the greatest.

These conditions are also sufficient. *Illustration:*

	Choice			
	1st	2d	3d	
A	9	2	9	
B	4	11	5	
C	7	7	6	Winner
Weights	4	2	1	A
	6	5	1	B
	3	2	1	C

A DIRECT PROOF OF DE MOIVRE'S FORMULA.

By S. LEFSCHETZ, University of Kansas.

The proposition which will be proved here, and which is practically equivalent to De Moivre's noted theorem, may be stated thus:

If X , Y , Z are three complex numbers of modulus unity, and such that their arguments x , y , z have a zero sum, then $XYZ = 1$.

Let $X = a + ia'$, $Y = b + ib'$, $Z = c + ic'$, and denote their conjugates by $\bar{X}$, $\bar{Y}$, $\bar{Z}$. The moduli being unity, we have $\bar{X} = 1/X$, $\bar{Y} = 1/Y$, $\bar{Z} = 1/Z$. Let A and B be the representative points of X and $\bar{Y}$, using Argand's diagram, O the origin, E the foot of the perpendicular AE from A to OB . As angle $BOA = (x + y)$, it follows that $OE = -c$, $EA = -c'$. The coefficients of the equations of the lines OB and AE are rational in the coördinates of A , B , that is, in a , b , a' , b' , and therefore the same holds for the coördinates of their intersection E . When these are known, the ratio $OE/OB = -c/1$ is determined rationally. Hence c , and similarly c' , can be determined rationally with respect to a , b , a' , b' . Therefore.

$$(1) \quad Z = c + ic' = f(a, b, a', b'),$$

where f is a rational function. But

$$a = \frac{1}{2}(X + \bar{X}) = \frac{1}{2}\left(X + \frac{1}{X}\right), \quad a' = \frac{1}{2i}\left(X - \frac{1}{X}\right), \quad b = \frac{1}{2}\left(Y + \frac{1}{Y}\right),$$

$$b' = \frac{1}{2i}\left(Y - \frac{1}{Y}\right).$$

Substituting in (1), it can be replaced by a relation such as

$$(2) \quad F(X, Y, Z) = 0,$$

where F is a polynomial in X, Y, Z , of the first degree in Z . When the three variables are arbitrarily permuted, five other relations of the same type are obtained which, for reasons of symmetry, must be equally satisfied. But between X, Y, Z , only one relation can exist, since two of them are arbitrary. Hence the polynomials at the left of the relations analogous to (2), which have just been considered, have a common divisor $\phi(X, Y, Z)$ symmetric with respect to X, Y, Z , and we have $\phi = 0$. As ϕ , like F itself, is necessarily of the first degree in Z , it is linear in the three variables and we have a relation such as

$$(3) \quad A \cdot XYZ + B(XY + YZ + ZX) + C(X + Y + Z) + D = 0,$$

A, B, C being coefficients still to be determined.

If the arguments are changed in sign, X, Y, Z are changed into their conjugates, and (3) is still satisfied. Hence A, B, C, D are real. As the conjugates are also the reciprocals in this case, coefficients in (3) equidistant from the extremes can differ only in sign.

Hence $D = kA, C = kB, k = \pm 1$. But for $x = y = \pi/2$, we have $z = -\pi$, $X = Y = i, Z = -1$. Therefore,

$$A(1 + k) + B[-1 - 2i + k(2i - 1)] = 0.$$

Equating to zero real and complex parts respectively, we obtain

$$(1 + k)(A - B) = 0, \quad (1 - k)B = 0.$$

Hence, either: (a) $k = +1, A = B$; or (b) $k = -1, B = 0$, when we can take $A = 1$. But if $x = y = z = 0$, then $X = Y = Z = 1$. Substituting in (3), we have

$$(A + 3B)(1 + k) = 0,$$

which is incompatible with assumption (a).

Hence $k = -1, B = 0$, and therefore $XYZ = 1$, as was to be proved.

Since

$$X = \cos x + i \sin x, \quad Y = \cos y + i \sin y, \quad z = -(x + y),$$

we have

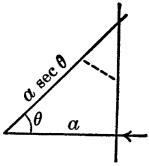
$$\begin{aligned}(\cos x + i \sin x)(\cos y + i \sin y) &= 1/Z = \bar{Z} = \cos(-x-y) - i \sin(-x-y) \\ &= \cos(x+y) + i \sin(x+y),\end{aligned}$$

which is De Moivre's formula. From this the trigonometric addition or subtraction formulas are easily derived in all their generality.

SAILING TO WINDWARD.

BY W. E. BYERLY, Harvard University.

The problem of sailing to a windward goal is usually the simple problem of *getting to windward*. For, if we find the straight path of least time from the boat to a perpendicular to the wind's direction, by tacking at the proper point we shall reach any goal in that perpendicular in the time it would take to reach the perpendicular had we continued on the original course.



The velocity of the boat on any course is some function of θ depending on the model and sailing qualities of the boat, where θ is the angle which the course makes with the direction of the wind, and we shall suppose this function given, and shall represent it by $F(\theta)$.

The time required to get the distance a to windward is the distance sailed, $a \sec \theta$, divided by the velocity, $F(\theta)$, and we wish to make this time a minimum.

Let $u = a \sec \theta / F(\theta)$. Then

$$\frac{du}{d\theta} = \frac{F(\theta) a \sec \theta \tan \theta - a \sec \theta F'(\theta)}{[F(\theta)]^2} = 0, \quad \text{where} \quad F'(\theta) = \frac{d}{d\theta} F(\theta),$$

or

$$(1) \quad F(\theta) \tan \theta - F'(\theta) = 0.$$

The solution of this equation will give us the angle which our course on each tack should make with the direction of the wind, whether we are using sails alone or sails and auxiliary motor.

In any concrete case equation (1) can be solved by "trial and error" with the aid of a three-place trigonometric table (preferably giving the angles in radians as well as in degrees) with sufficient accuracy for all practical purposes.

The function $F(\theta)$ ought to be determined by experiment for every boat, but as a first approximation, in the case of a sail boat, it may be taken as equal to $k(\theta - \alpha)$, where α , "the angle of repose," is the angle with the direction of the wind within which the boat will not sail; an angle, by the way, that is very easily discovered by experimenting with the boat.